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On the definition of the classical connectives and quantifiers

Gilles Dowek

INRIA, 23 avenue d'Italie, CS 81321, 75214 Paris Cedex 13, France.
`gilles.dowek@inria.fr`

The history of the notion of constructivity started with a dispute on the deduction rules that one should or should not use to prove a theorem. Depending on the rules accepted by the ones and the others, the proposition $P \vee \neg P$, for instance, had a proof or not. A less controversial situation was reached with a classification of proofs, and it became possible to agree that this proposition had a classical proof but no constructive proof.

An alternative is to use the idea of Hilbert and Poincaré that axioms and deduction rules define the meaning of the symbols of the language and it is then possible to explain that some judge the proposition $P \vee \neg P$ true and others do not because they do not assign the same meaning to the symbols \vee , \neg , etc. The need to distinguish several meanings of a common word is usual in mathematics. For instance the proposition “there exists a number x such that $2x = 1$ ” is true or false depending on whether the word “number” means “natural number” or “real number”. Even for logical connectives, the word “or” has to be disambiguated into inclusive and exclusive.

Taking this idea seriously, we should not say that the proposition $P \vee \neg P$ has a classical proof but no constructive proof, but we should say that the proposition $P \vee^c \neg^c P$ has a proof, but not the proposition $P \vee \neg P$, that is we should introduce two symbols for each connective and quantifier, for instance a symbol \vee for the constructive disjunction and a symbol \vee^c for the classical one, instead of introducing two judgements: “has a classical proof” and “has a constructive proof”. We should also be able to address the question of the provability of mixed propositions and, for instance, express that the proposition $(\neg(P \wedge Q)) \Rightarrow (\neg P \vee^c \neg Q)$ has a proof.

The idea that the meaning of connectives and quantifiers is expressed by the deduction rules leads to propose a logic containing all the constructive and classical connectives and quantifiers and deduction rules such that a proposition containing only constructive connectives and quantifiers has a proof in this logic if and only if it has a proof in constructive logic and a proposition containing only classical connectives and quantifiers has a proof in this logic if and only if it has a proof in classical logic. Such a logic containing classical, constructive, and also linear, connectives and quantifiers has been proposed by J.-Y. Girard [3]. This logic is a sequent calculus with *unified sequents* that contain a linear zone and a classical zone and rules treating differently propositions depending on the zone they belong.

Our goal in this paper is slightly different, as we want to define the meaning of a small set of primitive connectives and quantifiers with deduction rules and

define the others explicitly, in the same way the exclusive or is explicitly defined in terms of conjunction, disjunction and negation: $A \oplus B = (A \wedge \neg B) \vee (\neg A \wedge B)$. A first step in this direction has been made by Gödel [4] who defined a translation of constructive logic into classical logic, and Kolmogorov [7], Gödel [5], and Gentzen [2] who defined a translation of classical logic into constructive logic. As the first translation requires a modal operator, we shall focus on the second. This leads to consider constructive connectives and quantifiers as primitive and search for definitions of the classical ones. Thus, we want to define classical connectives and quantifiers \top^c , \perp^c , \neg^c , \wedge^c , \vee^c , \Rightarrow^c , \forall^c , and \exists^c and embed classical propositions into constructive logic with a morphism $\| \cdot \|$ defined as follows.

Definition 1 (Morphism).

- $\|P\| = P$ if P is an atomic proposition
- $\|\top\| = \top^c$
- $\|\perp\| = \perp^c$
- $\|\neg A\| = \neg^c \|A\|$
- $\|A \wedge B\| = \|A\| \wedge^c \|B\|$
- $\|A \vee B\| = \|A\| \vee^c \|B\|$
- $\|A \Rightarrow B\| = \|A\| \Rightarrow^c \|B\|$
- $\|\forall x A\| = \forall^c x \|A\|$
- $\|\exists x A\| = \exists^c x \|A\|$

If $\Gamma = A_1, \dots, A_n$ is a multiset of propositions, we write $\|\Gamma\|$ for the multiset $\|A_1\|, \dots, \|A_n\|$.

Kolmogorov-Gödel-Gentzen translation can be defined as follows

- $(P)' = \neg\neg P$, if P is an atomic proposition
- $(\top)' = \neg\neg\top$
- $(\perp)' = \neg\neg\perp$
- $(\neg A)' = \neg\neg\neg(A)'$
- $(A \wedge B)' = \neg\neg((A)' \wedge (B)')$
- $(A \vee B)' = \neg\neg((A)' \vee (B)')$
- $(A \Rightarrow B)' = \neg\neg((A)' \Rightarrow (B)')$
- $(\forall x A)' = \neg\neg(\forall x (A)')$
- $(\exists x A)' = \neg\neg(\exists x (A)')$

or more succinctly as

- $(P)' = \neg\neg P$, if P is an atomic proposition
- $(*)' = \neg\neg*$, if $*$ is a zero-ary connective
- $(*A)' = \neg\neg(* (A)'),$ if $*$ is a unary connective
- $(A * B)' = \neg\neg((A)' * (B)'),$ if $*$ is a binary connective
- $(*x A)' = \neg\neg(*x (A)'),$ if $*$ is a quantifier

For instance

$$(P \vee \neg P)' = \neg\neg(\neg\neg P \vee \neg\neg\neg\neg P)$$

And it is routine to prove that a proposition A has a classical proof if and only if the proposition $(A)'$ has a constructive one.

This translation does not directly provide definitions of classical connectives and quantifiers because it is not a morphism. For instance, if we take the definition

- $\neg^c A = \neg\neg A$
- $A \vee^c B = \neg\neg(A \vee B)$
- etc.

where a double negation is put before each connective and quantifier, then the proposition $P \vee^c \neg^c P$ is $\neg\neg(P \vee \neg\neg P)$ where the double negations in front of atomic propositions are missing. Thus, we would need to also introduce a proposition symbol P^c defined by $P^c = \neg\neg P$. But this would lead us too far: we want to introduce constructive and classical versions of the logical symbols—the connectives and the quantifiers—but not of the non logical ones, such as the predicate symbols.

Another translation introduced by L. Allali and O. Hermant [1] leads to the definition

- $\neg^c A = \neg\neg A$
- $A \vee^c B = (\neg\neg A) \vee (\neg\neg B)$
- etc.

where double negations are put after, and not before, each connective and quantifier.

The proposition $P \vee^c \neg^c P$ is then $\neg\neg P \vee \neg\neg\neg\neg P$ where the double negation at the top of the proposition is missing. Using this translation Allali and Hermant prove that the proposition A has a classical proof if and only if the proposition $\neg\neg\|A\|$ has a constructive one and they introduce another provability judgement expressing that the proposition $\neg\neg A$ has a constructive proof. This also would lead us too far: in our logic, we want a single judgement “ A has a proof” expressing that A has a constructive proof, and not to introduce a second judgement, whether it be “ A has a classical proof” or “ $\neg\neg A$ has a proof”.

In order to do so, we define the classical connectives and quantifiers by introducing double negations both before and after each symbol.

Definition 2 (Classical connectives and quantifiers).

- $\top^c = \neg\neg\top$
- $\perp^c = \neg\neg\perp$
- $\neg^c A = \neg\neg\neg\neg A$
- $A \wedge^c B = \neg\neg((\neg\neg A) \wedge (\neg\neg B))$
- $A \vee^c B = \neg\neg((\neg\neg A) \vee (\neg\neg B))$
- $A \Rightarrow^c B = \neg\neg((\neg\neg A) \Rightarrow (\neg\neg B))$
- $\forall^c x A = \neg\neg(\forall x (\neg\neg A))$
- $\exists^c x A = \neg\neg(\exists x (\neg\neg A))$

Notice that the propositions $\top \Leftrightarrow \top^c$, $\perp \Leftrightarrow \perp^c$, and $\neg A \Leftrightarrow \neg^c A$ where $A \Leftrightarrow B$ is defined as $(A \Rightarrow B) \wedge (B \Rightarrow A)$, have proofs. Thus, the symbols \top^c , \perp^c , and \neg^c could be just defined as \top , \perp , and \neg .

With this definition, neither the double negations in front of atomic propositions nor those at the top of the proposition are missing. The price to pay is to have four negations instead of two in many places, but this is not harmful.

Yet, there is still a problem with the translation of atomic propositions: as with any morphism, the proposition P alone is translated as P and not as $\neg\neg P$. Thus, the property that a sequent $\Gamma \vdash A$ has a classical proof if and only if the sequent $\|\Gamma\| \vdash \|A\|$ has a constructive one only holds when A is not atomic. For instance, the sequent $P \wedge^c Q \vdash P$, that is $\neg\neg((\neg\neg P) \wedge (\neg\neg Q)) \vdash P$, does not have a constructive proof.

A solution to this problem is to decompose hypothetical provability into absolute provability and entailment. For absolute provability, the property that a sequent $\vdash A$ has a classical proof if and only if the sequent $\vdash \|A\|$ has a constructive one holds for all propositions, because atomic propositions have no proof. Thus, the sequent $H_1, \dots, H_n \vdash A$ has a classical proof if and only if the sequent $\vdash \|H_1\| \Rightarrow^c \dots \Rightarrow^c \|H_n\| \Rightarrow^c \|A\|$ has a constructive one. This leads to a system where we have only one notion of absolute provability, but two notions of entailment: “ A has a proof from the hypothesis H ” can either be understood as “ $H \Rightarrow A$ has a proof” or “ $H \Rightarrow^c A$ has a proof”.

| | |
|--|--|
| $\overline{A \vdash A}$ axiom | |
| $\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta} \text{contr-l}$ | $\frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta} \text{contr-r}$ |
| $\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{weak-l}$ | $\frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{weak-r}$ |
| | $\Gamma \vdash \top, \Delta \top\text{-r}$ |
| $\frac{\Gamma, \perp \vdash \Delta}{\Gamma \vdash A, \Delta} \perp\text{-l}$ | |
| $\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta} \neg\text{-l}$ | $\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta} \neg\text{-r}$ |
| $\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge\text{-l}$ | $\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta} \wedge\text{-r}$ |
| $\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta} \vee\text{-l}$ | $\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee\text{-r}$ |
| | $\frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta} \vee\text{-r}$ |
| $\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta} \Rightarrow\text{-l}$ | $\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \Rightarrow\text{-r}$ |
| $\frac{\Gamma, (t/x)A \vdash \Delta}{\Gamma, \forall x A \vdash \Delta} \forall\text{-l}$ | $\frac{\Gamma \vdash A, \Delta}{\Gamma \vdash \forall x A, \Delta} \forall\text{-r}$ |
| $\frac{\Gamma, A \vdash \Delta}{\Gamma, \exists x A \vdash \Delta} \exists\text{-l}$ | $\frac{\Gamma \vdash (t/x)A, \Delta}{\Gamma \vdash \exists x A, \Delta} \exists\text{-r}$ |

Fig. 1. Sequent calculus

Definition 3 (Classical and constructive provability). *Classical provability is defined by the cut free sequent calculus rules of Figure 1. We say that the proposition A has a classical proof if the sequent $\vdash A$ does.*

Constructive provability, our main notion of provability, is obtained by restricting to sequents with at most one conclusion. This requires a slight adaptation of the \Rightarrow -l rule

$$\frac{\Gamma \vdash A \quad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta} \Rightarrow\text{-l}$$

We say that the proposition A has a constructive proof if the sequent $\vdash A$ does.

Proposition 1. *If the proposition $\|A\|$ has a constructive proof, then the proposition A has a classical one.*

Proof. If the proposition $\|A\|$ has a constructive proof, then $\|A\|$ also has a classical proof. Hence, $\|A\|$ being classically equivalent to A , A also has a classical proof.

We now want to prove the converse: that if the proposition A has a classical proof, then $\|A\|$ has a constructive proof. To do so, we first introduce another translation where the top double negation is removed, when there is one.

Definition 4 (Light translation).

- $|P| = P$,
- $|\top| = \top$,
- $|\perp| = \perp$,
- $|\neg A| = \neg\neg\|A\|$,
- $|A \wedge B| = (\neg\neg\|A\|) \wedge (\neg\neg\|B\|)$,
- $|A \vee B| = (\neg\neg\|A\|) \vee (\neg\neg\|B\|)$,
- $|A \Rightarrow B| = (\neg\neg\|A\|) \Rightarrow (\neg\neg\|B\|)$,
- $|\forall x A| = \forall x (\neg\neg\|A\|)$,
- $|\exists x A| = \exists x (\neg\neg\|A\|)$.

If $\Gamma = A_1, \dots, A_n$ is a multiset of propositions, we write $|\Gamma|$ for the multiset $|A_1|, \dots, |A_n|$ and $\neg|\Gamma|$ for the multiset $\neg|A_1|, \dots, \neg|A_n|$.

Proposition 2. *If the proposition A is atomic, then $\|A\| = |A|$, otherwise $\|A\| = \neg\neg|A|$.*

Proof. By a case analysis on the form of the proposition A .

Proposition 3. *If the sequent $\Gamma, |A| \vdash$ has a constructive proof, then so does the sequent $\Gamma, \|A\| \vdash$.*

Proof. By Proposition 2, either $\|A\| = |A|$, or $\|A\| = \neg\neg|A|$. In the first case the result is obvious, in the second, we build a proof of $\Gamma, \|A\| \vdash$ with a \neg -l rule, a \neg -r rule, and the proof of $\Gamma, |A| \vdash$.

Proposition 4. *If the sequent $\Gamma \vdash \Delta$ has a classical proof, then the sequent $|\Gamma|, \neg|\Delta| \vdash$ has a constructive one.*

Proof. By induction on the structure of the classical proof of the sequent $\Gamma \vdash \Delta$. As all the cases are similar, we just give a few.

- If the last rule is the *axiom* rule, then $\Gamma = \Gamma', A$ and $\Delta = \Delta', A$, and the sequent $|\Gamma'|, |A|, \neg|A|, \neg|\Delta| \vdash$, that is $|\Gamma|, \neg|\Delta| \vdash$, has a constructive proof.
- If the last rule is the \Rightarrow -*l* rule, then $\Gamma = \Gamma', A \Rightarrow B$ and by induction hypothesis, the sequents $|\Gamma'|, \neg|A|, \neg|\Delta| \vdash$ and $|\Gamma'|, |B|, \neg|\Delta| \vdash$ have constructive proofs, thus the sequents $|\Gamma'|, \neg\|A\|, \neg|\Delta| \vdash$ and $|\Gamma'|, \|B\|, \neg|\Delta| \vdash$ have constructive proofs, thus, using Proposition 3, the sequent $|\Gamma'|, \neg\|A\| \Rightarrow \neg\|B\|, \neg|\Delta| \vdash$, that is $|\Gamma|, \neg|\Delta| \vdash$, has a constructive proof.
- If the last rule is the \Rightarrow -*r* rule, then $\Delta = \Delta', A \Rightarrow B$ and by induction hypothesis, the sequent $|\Gamma|, |A|, \neg|B|, \neg|\Delta'| \vdash$ has a constructive proof, thus, using Proposition 3, the sequent $|\Gamma|, \|A\|, \neg\|B\|, \neg|\Delta'| \vdash$ has a constructive proof, thus the sequent $|\Gamma|, \neg(\neg\|A\| \Rightarrow \neg\|B\|), \neg|\Delta'| \vdash$, that is $|\Gamma|, \neg|\Delta| \vdash$, has a constructive proof.

Proposition 5. *If the sequent $\Gamma \vdash A$ has a classical proof and A is not an atomic proposition, then the sequent $\|\Gamma\| \vdash \|A\|$ has a constructive one.*

Proof. By Proposition 4, as the sequent $\Gamma \vdash A$ has a classical proof, the sequent $|\Gamma|, \neg|A| \vdash$ has a constructive one. Thus, by Proposition 3, the sequent $\|\Gamma\|, \neg|A| \vdash$ has a constructive proof, and the sequent $\|\Gamma\| \vdash \neg\neg|A|$ also. By Proposition 2, as A is not atomic, $\|A\| = \neg\neg|A|$. Thus, the sequent $\|\Gamma\| \vdash \|A\|$ has a constructive proof.

Theorem 1. *The proposition A has a classical proof if and only if the proposition $\|A\|$ has a constructive one.*

Proof. By Proposition 1, if the proposition $\|A\|$ has a constructive proof, then the proposition A has a classical one. Conversely, we prove that if the proposition A has a classical proof, then the proposition $\|A\|$ has a constructive one. If A is atomic, the proposition A does not have a classical proof, otherwise, by Proposition 5, the proposition $\|A\|$ has a constructive proof.

Corollary 1. *The sequent $H_1, \dots, H_n \vdash A$ has a classical proof if and only if the sequent $\vdash \|H_1\| \Rightarrow^c \dots \|H_n\| \Rightarrow^c \|A\|$ has a constructive one.*

Proof. The sequent $H_1, \dots, H_n \vdash A$ has a classical proof if and only if the sequent $\vdash H_1 \Rightarrow \dots H_n \Rightarrow A$ has one and, by Theorem 1, if and only if the sequent $\vdash \|H_1\| \Rightarrow^c \dots \|H_n\| \Rightarrow^c \|A\|$ has a constructive proof.

There is no equivalent of Theorem 1 if we add double negations after the connectors only. For instance, the proposition $P \vee \neg P$ has a classical proof, but the proposition $\neg\neg P \vee \neg\neg\neg\neg P$ has no constructive proof. O. Hermant [6] has proved that there is also no equivalent of Theorem 1 if we add double negations before the connectors only. For instance, the proposition $(\forall x (P(x) \wedge Q)) \Rightarrow (\forall x P(x))$ has a classical proof, but the proposition $\neg\neg((\neg\neg\forall x \neg\neg(P(x) \wedge Q)) \Rightarrow (\neg\neg\forall x P(x)))$ has no constructive proof.

In many cases, however, even the proposition $H \Rightarrow \|A\|$ has a constructive proof. For instance, consider the theory formed with the axiom H “The union of two finite sets is finite”

—or, as the cut rule is admissible in sequent calculus, any theory where this proposition has a proof—and let A be the proposition “If the union of two sets is infinite then one of them is”

which is, for instance, at the heart of the proof of Bolzano-Weierstrass theorem, then the proposition $H \Rightarrow \|A\|$ has a constructive proof

where the only classical connective is the disjunction, has a constructive proof.

Which mathematical results have a classical formulation that can be proved from the axioms of constructive set theory or constructive type theory and which require a classical formulation of these axioms and a classical notion of entailment remains to be investigated.

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